MACM 101 : Tutorial  
October 22, 2018

1. (9 points) Consider the following theorem: If $a$ is an odd number, $a^2 + 2a + 5$ is even.

(a) Give a direct proof of the theorem.

Suppose $a$ is an odd number, $a^2$ is odd, $2a$ is even, $5$ is odd, thus $a^2 + 2a + 5$ is even.

(b) Give a contrapositive proof of the theorem.

Suppose $a^2 + 2a + 5$ is odd, we know $2a$ is even, $5$ is odd, so $a^2$ is even. Then $a$ is even.

(c) Give a proof by contradiction of the theorem.

$a$ is an odd number, suppose $a^2 + 2a + 5$ is odd, then we have $a^2$ is odd, $2a$ is even. Then $5$ must be an even number in order to make $a^2 + 2a + 5$ odd. We have a contradiction since $5$ is odd number.

2. (5 points) Prove the following theorem.

- For all positive integers $n$: $n$ is even if and only if $3n^2 + 8$ is even.

Suppose $n$ is even, then $n^2$ is even, $3n^2$ is even, $8$ is even, so $3n^2 + 8$ is even.

Suppose $3n^2 + 8$ is even, we know $8$ is even, so $3n^2$ is even. Thus $n$ is even.

- Given an integer $n$, then $n^3 + n^2 + n$ is even if and only if $n$ is even.

Suppose $n$ is even, then $n^3$, $n^2$, all even. Thus $n^3 + n^2 + n$ is even.

Suppose $n^3 + n^2 + n$ is even, since $n^3$, $n^2$, $n$ are of the same parity. Then $n$ is even.

- Prove that $\sqrt{2}$ is irrational.

Proof by contradiction Suppose $\sqrt{2}$ is rational. Then $\sqrt{2} = a/b$ for some integer $a,b$ where $a$ and $b$ have no common factors and $b$ is non zero.
By squiring both sides of the equation we have $a^2 = 2b^2$
Then $a$ is even. Suppose $a = 2k$ for some integer $k$
Then we have $4k^2 = 2b^2$, Thus $b = 2k^2$, Then $b$ is even.
We derive a contradiction that $a$ and $b$ have common factor 2.
Thus $\sqrt{2}$ is irrational.

3. (5 points) Show that the following statements are equivalent.

- $p_1 : n$ is an even integer
- $p_2 : (n + 1)$ is an odd integer
- $p_3 : n^2$ is an even integer.

If $n$ is an even integer, then $n + 1$ is an odd integer. If $n + 1$ is an odd integer, $n$ is even. So $p_1$ and $p_2$ are equivalent.
If $n$ is an even integer, then $n^2$ is an even integer. If $n^2$ is an even integer, $n$ is even. So $p_1$ and $p_3$ are equivalent.
So $p_1, p_2, p_3$ are equivalent.

4. (3 points) Disprove by counterexample the following proposition.

- The product of two irrational numbers is always irrational.

$\sqrt{2}$ is irrational number, and $\sqrt{2}^2 = 2$ which is rational.
- If $x, y \in \mathbb{R}$, then $|x + y| = |x| + |y|$.

Let $x = 1$ and $y = -1$. Then $|x + y| = 0$ but $|x| + |y| = 2$

5. (Bonus Question) (5 points)

- If $p$ is a prime and $0 < k < p$, $p | \binom{p}{k}$.

We know $\frac{p!}{k!(p-k)!} = \binom{p}{k} = \frac{1 \times 2 \times \ldots \times p}{1 \times 2 \times \ldots \times k \times 1 \times 2 \times \ldots \times (p-k)}$, since $k < p$, all the factors in the denominator less than $p$, so they do not cancel the $p$ in the numerator. So $p | \binom{p}{k}$.
- If $n \in \mathbb{N}$, $\binom{2n}{n}$ is even.

We are asked to choose $N$ items from a set of $2N$ items. Suppose the items we choose form a set $A$. Then $2N - A$ is also a solution. Thus the solution comes in pairs. $\binom{2n}{n}$ is even.