Part A: Answer questions worth 40 points

1. (10 points) Consider a deck of 40 cards consists of cards numbered 1, 2, \ldots , 10 in red, yellow, green and blue. Find the number of ways one can select 5-card sequence when

(a) all sequences with no card restrictions

Solution: \( P(40, 5) \) (order is important)

(b) all sequences with cards all the same colour

Solution: For red coloured cards, there are \( P(10, 5) \) 5-card sequences. The answer is \( 4 \times P(10, 5) \) for all 5 colors.

(c) all sequences with two or more cards having the same number

Solution: Let us count all 5-card sequences where all the cards have distinct numbers. The number of such sequences is \( 40 \times 36 \times 32 \times 28 \times 24 \). After selecting the first card, we are not allowed to pick any card with the same number as the first card. We use the same strategy to select the remaining cards. The answer to the question is, therefore, \( P(40, 5) - 40 \times 36 \times 32 \times 28 \times 24 \).

(d) all sequences that start with 2 and end with 8

Solution: There are \( P(4, 1) \) (or \( C(4, 1) \)) ways to select 2 and \( P(4, 1) \) ways to select 8. The other three cards could be anything. The answer is, therefore, \( P(4, 1)^2 \cdot P(38, 3) \).

2. (10 points) Consider the 13-letter word MASSACHUSETTS.

(a) Determine the number of different strings of length 13 that can be formed from all the letters of the word.

Solution: In MASSACHUSETTS, there are 1 M, 2 As, 4 Ss, 1 C, 1H, 1 H, 1 E and 2 Ts. The number of strings that can be formed is \( \frac{13!}{2!4!2!} \).
(b) Determine the number of different strings that can be formed from all the letters of the word where no two S’s can appear side by side.

**Solution:** We first note that there are \( \frac{9!}{2! \cdot 2!} \) strings that do not contain S. We now insert 4 Ss to each string of 9 letters. Since no two Ss can appear together, 4 Ss have to be inserted in 10 positions determined by a 9-letter string. There are \( \binom{10}{4} \) ways one can determine 4 positions for Ss. Therefore, the answer is \( \binom{10}{4} \times \frac{9!}{2! \cdot 2!} \).

(c) Determine the number of different strings that can be formed from all the letters of the word where C and E cannot be side by side (in either order).

**Solution:** Using arguments similar to (b) we can show that the number of strings satisfying the constraints is \( P(12,2) \times \frac{8!}{3! \cdot 2! \cdot 2!} \).

3. (5 points) In the expansion of \((1 + x)^8 - (1 - 2x)^7\), determine the coefficient of \(x^5\).

**Solution:** The coefficient of \(x^5\) in the binomial expansion of \((1 + x)^8\) is \( \binom{8}{5}.(1)^5 \). The coefficient of \(x^5\) in the binomial expansion of \((1 - 2x)^7\) is \( \binom{7}{5}.(-2)^5 \). The answer is, therefore \( \binom{8}{5}.\binom{7}{5}.2^5 \).

4. (10 points) Using Laws of Logic show that \(\neg p \land \neg q\) and \(\neg (p \lor (\neg p \land q))\) are logically equivalent.

**Solution**
\[
\begin{align*}
(p \lor (\neg p \land q)) & \iff (\neg p \land (\neg p \lor q)) & \text{DeMorgan’s law} \\
& \iff (\neg p \land (p \lor q)) & \text{DeMorgan’s law} \\
& \iff (\neg p \land (p \lor q)) & \text{Law of double negation} \\
& \iff (\neg p) \lor (\neg p \land q) & \text{Distributed law} \\
& \iff \neg p \lor q & \text{Inverse law} \\
& \iff \neg p \land q & \text{Identity law}
\end{align*}
\]

5. (10 points) Write the following arguments in symbolic form. Then establish the validity of the argument.

If Sandy gets the manager’s position and works hard, then she will get a raise. If she gets the raise, she will buy a new truck. She has not purchased a new truck. Therefore either Sandy did not get the manager’s position or she did not work hard.

**Solution:** Symbolic equivalence:

\( p: \) Sandy gets manager’s position;  
\( q: \) Sandy works hard;
r: sandy gets a raise;  
s: Sandy buys a new car;  

The arguments are:  
\((p \land q) \rightarrow r\)  
r \rightarrow s  
\neg s  

\neg p \lor \neg q  

Validity of the argument  

(1) \neg s \quad \text{Premise}  
(2) r \rightarrow s \quad \text{Premise}  
(3) \neg r \quad \text{Steps (1), (2) and Modus Tollens}  
(4) (p \land q) \rightarrow r \quad \text{Premise}  
(5) \neg(p \land q) \quad \text{Steps (3), (4) and Modus Tollens}  
(6) \therefore \neg p \lor \neg q \quad \text{Step (5) and DeMorgan’s law}  

6. (5 points) Consider the open statements:  

- \(p(x) : x > 0\)  
- \(q(x) : x^2 \geq 0\)  
- \(r(x) : (x - 4)(x + 1) = 0\)  

Determine the truth value of the following statements, where the universe (domain) of discourse of each variable is the set of real numbers (positive and negative).  

(a) \(\exists x \ [p(x) \land q(x)]\) \quad \text{TRUE when } x=2  
(b) \(\forall x \ [p(x) \rightarrow q(x)]\) \quad \text{TRUE for any positive } x  
(c) \(\forall x \ [r(x) \rightarrow p(x)]\) \quad \text{False. When } x=-1, r(x) \text{ is true, but } p(x) \text{ is false.}  

Part B: Bonus Question (10 points)  

Determine the number of integral solutions to the following equation:  
\[ x_1 + x_2 + x_3 + x_4 + x_5 = 50, \quad 1 \leq x_i, \quad \forall i, \]  

where for each \(i, 1 \leq i \leq 5, x_i \text{ is a multiple of } 5.\)  

Solution: Suppose we write \(x_i = 5y_i, i = 1,2,3,4,5.\) The problem can then be restated as:
Determine the number of integral solutions to the following equation:

$$5y_1 + 5y_2 + 5y_3 + 5y_4 + 5y_5 = 50, \quad 1 \leq y_i, \quad \forall i.$$ 

which is the same as

$$y_1 + y_2 + y_3 + y_4 + y_5 = 10, \quad 1 \leq y_i, \quad \forall i.$$ 

The answer is $C(5 + 5 - 1, 5 - 1) = C(9, 4)$. 
