1. Answer the following short questions.

(a) For which sets \( A, B \) is it true that \( A \times B = B \times A \)?

Solution: When one of the sets is empty. It is also true when \( A = B \).

(b) If \( A = \{1, 2, 3, 4, 5\} \) and \( B = \{w, x, y, z\} \), how many elements are in \( \mathcal{P}(A \times B) \)?

Solution: \( 2^{|A \times B|} \).

(c) Let \( f : \mathbb{Z}^+ \to \mathbb{Z}^+ \) where for all \( x \in \mathbb{Z}^+ \), \( f(x) = x + 1 \). What is the range of \( f \)? Is \( f \) one-to-one? Is it onto?

Solution: The range of \( f \) is \( \{2, 3, 4, \ldots\} \). \( f \) is one-to-one, but not onto. \( f^{-1}(1) \) is not defined.

(d) Let \( A = \{a, b, c\} \). Define the function \( f : \mathcal{P}(A) \to \mathbb{N} \) by \( f(X) = |X| \). Find the range of \( f \).

Solution: The range of \( f \) is the sizes of all valid subsets of \( A \), which is \( \{0, 1, 2, 3\} \).

(e) How many times must we roll a single die in order to get the same score \( n \) times.

Solution: After 7 rolls, we are guaranteed to see a score at least twice. To see a score at least \( n \) times, in the worst case, we need to roll the dice \( 6(n - 1) + 1 \) times.

2. (5 points) Over a 44 day period, Gary will train for triathlons at least once per day, and a total of 70 times in all. Show that there is a period of consecutive days during which he trains exactly 17 times.

Solution: Let \( a_i \) be the number of games Garry played from day 1 to day \( i \) altogether. Therefore, \( 1 \leq a_1 \leq a_2 \leq \ldots \leq a_{44} \leq 70 \). Let \( b_i = a_i + 17 \).

Now define a function \( f : X \to Y \) where

- \( X = \{a_1, a_2, \ldots, a_{44}, b_1, b_2, \ldots, b_{44}\} \) and
- \( Y = \{1, 2, 3, \ldots, 87\} \) (the value each \( a_j \) or \( b_j \) is at least 1 and at most 87)
- \( f(x) = \) the value of \( x \).
- There are 88 pigeons and 87 holes. Hence there exists \( j \) and \( j \) such that \( a_j = b_j \). Thus during days \( j + 1, j + 2, \ldots, j \) Garry trains 17 games.

3. (5 points) Over a 30 day period, Rick will walk the dog at least once per day, and a total of 45 times in all. Prove that there is a period of consecutive days in which he walks the dog exactly 14 times.

Solutions: Here \( 1 \leq a_i \leq 45; 15 \leq b_i = a_i + 14 \leq 59 \). The rest of the arguments remain the same.
4. (5 points) Ten baseball teams are entered in a round-robin tournament (meaning that every team plays every other team exactly once) in which ties are not allowed. Prove that if no team loses all of its games, then some two teams finish the tournament with the same number of wins.

Solution: A problem is solved in the class. Let $f(i)$ be the number of wins for team $i$. Therefore, $1 \leq f(i) \leq 9$. Since the size of the domain of $f$ is 10 and the codomain size of $f$ is 9, there exist $i, j$ such that $f(i) = f(j)$.

5. (5 points) An auditorium has a seating capacity 800. How many seats must be occupied to guarantee that at least two people seated in the auditorium have the same first and last initials.

Solution: Here we assume that all initials are upper case. There are $26^2$ possible ways to assign the initials. The number of people in the auditorium should be $26^2 + 1$ at least to guarantee two people have fist and the second initials sitting together.

6. (5 points) Given 8 Perl books, 17 Visual Basic books, 6 Java books, 12 SQL books, and 20 C++ books, how many of these books must we select to ensure that we have 10 books dealing with the same computer language.

Solution: The answer is $(8 + 9 + 6 + 9 + 9) + 1$.

7. (Bonus Question) (5 points) Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two bijective functions. Show that $(g \circ f)^{-1}$ exists and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Solution: Let $z$ be an arbitrary element of $Z$. Interested in finding $w \in X$ such that $(g \circ f)^{-1}(z) = w$, i.e. $z = (g \circ f)(w)$. Now $(g \circ f)(w) = g(f(w))$. Since $f$ and $g$ are bijective, there exists $z' \in Y$ and $z'' \in X$ such that $g(z') = z$ and $f(z'') = z'$. Therefore, $(g \circ f)^{-1}(z) = z'' = w$. Hence $(g \circ f)^{-1}$ exists.

Second part: We just showed that for an arbitrary $z \in Z$, there exists $z' \in Y$ and $z'' \in X$ such that $(g \circ f)^{-1}(z'') = z'$, $f(z'') = z'$, and $g(z') = z$. Since $f$ and $g$ have inverses, $f^{-1}(z') = z''$, and $g^{-1}(z) = z'$. Now $f^{-1} \circ g^{-1}(z) = f^{-1}(g^{-1}(z)) = f^{-1}(z') = z''$.

8. prove that if $f$ is one-to-one, the $f(A \cup B) = f(A) \cup f(B)$ for any $A, B$ included in the domain of $f$.

Solution: We prove the statement by showing that $f(A \cap B) \subseteq f(A) \cap f(B)$ and $f(A) \cap f(B) \subseteq f(A \cap B)$.

- $f(A \cap B) \subseteq f(A) \cap f(B)$
  
  If $y \in f(A \cap B) \subseteq f(A) \cap f(B)$, then $y = f(x)$ with $x \in A \cap B$, i.e. $x \in A$ and $x \in B$. Therefore, $y \in f(A)$ and $y \in f(B)$. Thus $y \in f(A) \cap f(B)$.

- $f(A) \cap f(B) \subseteq f(A \cap B)$
  
  If $y \in f(A) \cap f(B)$, then $y = f(x)$ with $x \in A$ and $y = f(x')$ with $x' \in B$. Since $f$ is one-to-one and $f(x) = f(x')$, therefore $x = x'$. Hence $x \in A \cap B$. Thus $y = f(x) \in f(A \cap B)$. 

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